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Bounds for the trace norm of A_α matrix of digraphs

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ABSTRACT

Let D be a digraph of order n with adjacency matrix $A(D)$. For $\alpha \in [0,1)$, the A_α matrix of D is defined as $A_\alpha(D) = \alpha\Delta^+(D) + (1-\alpha)A(D)$, where $\Delta^+(D) = \text{diag}\ (d_1^+, d_2^+, \ldots, d_n^+)$ is the diagonal matrix of vertex out degrees of D. Let $\sigma_{1\alpha}(D), \sigma_{2\alpha}(D), \ldots, \sigma_{n\alpha}(D)$ be the singular values of $A_\alpha(D)$. Then the trace norm of $A_\alpha(D)$, which we call α trace norm of D, is defined as $\|A_\alpha(D)\|_* = \sum_{i=1}^n \sigma_{i\alpha}(D)$. In this paper, we find the singular values of some basic digraphs and characterize the digraphs D with Rank $(A_\alpha(D)) = 1$. As an application of these results, we obtain a lower bound for the trace norm of A_α matrix of digraphs and determine the extremal digraphs. In particular, we determine the oriented trees for which the trace norm of A_α matrix attains minimum. We obtain a lower bound for the α spectral norm $\sigma_{1\alpha}(D)$ of digraphs and characterize the extremal digraphs. As an application of this result, we obtain an upper bound for the α trace norm of digraphs and characterize the extremal digraphs.

1. Introduction

A directed graph (or briefly digraph) D consists of two sets $\mathcal{V}$ and $\mathcal{A}$, where $\mathcal{V}$ is a non-empty finite set whose elements are called vertices and $\mathcal{A}$ is a set of ordered pairs of elements of $\mathcal{V}$ and is known as set of arcs. We assume our digraphs are simple i.e., there are no loops and parallel arcs. A graph G can be identified with a symmetric digraph $\overleftrightarrow{G}$ obtained by replacing each edge e of G by a pair of symmetric arcs. We call a digraph to be asymmetric if it has no pair of symmetric arcs. An asymmetric digraph is also known as an oriented graph. In a digraph D, an arc from a vertex u to v is denoted by (u,v). In this case, we say u is the tail and v is the head of arc (u,v). The set of vertices $\{w \in \mathcal{V} : (u,w) \in \mathcal{A}\}$ is called the outer neighbour set of u and we denote it by $N^+(u)$. The set of vertices $\{w \in \mathcal{V} : (w,u) \in \mathcal{A}\}$ is called the inner neighbour set of u and we denote this by $N^-(u)$. The cardinality of the set $N^+(u)$ is called outdegree of u and we denote it by d_u^+. Therefore, $d_u^+ = |N^+(u)|$. The cardinality of the set $N^-(u)$ is called indegree of u and we denote it by d_u^-. Therefore, $d_u^- = |N^-(u)|$.

Let D be a digraph with vertex set $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$. Then the adjacency matrix $A(D) = (a_{ij})$ of D is a square matrix of order n with $a_{ij} = 1$ if there is an arc from vertex v_i to vertex v_j and zero, otherwise. Let $\Delta^+ = \Delta^+(D) = \text{diag}\ (d_1^+, d_2^+, d_3^+, \ldots, d_n^+)$, where $d_i^+ = d_{v_i}^+$, be the outdegree matrix of D. Then the Laplacian and signless Laplacian matrices of D are respectively defined as $L(D) = \Delta^+ - A(D)$ and $Q(D) = \Delta^+ + A(D)$. For $\alpha \in [0,1]$, Nikiforov [19] defined the α adjacency matrix $A_\alpha(G)$ of a graph G as a common extension of adjacency matrix and signless Laplacian matrix

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as $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$, where $D(G)$ is diagonal matrix of vertex degrees of G. The spectral properties of α matrix of graphs are well studied, for example see [14,15] and references cited therein. Liu et al. [13], defined the A_α matrix for directed graphs and studied α-spectral radius of digraphs. For $\alpha \in [0,1)$, the α matrix of a digraph is defined as $A_\alpha(D) = \alpha\Delta^+(D) + (1-\alpha)A(D)$. For $\alpha = 0$, we see $A_0 = A$ and for $\alpha = \frac{1}{2}$, $A_{\frac{1}{2}} = \frac{1}{2}Q(D)$. It is clear that $A_\alpha(D)$ is a common extension of adjacency matrix $A = A(D)$ and signless Laplacian matrix $Q(D)$ of a digraph D. Very recently, Yang et al. [24] studied the spectral moments of A_α adjacency matrix of a digraph. For α spectral radius see [5].

Let G be an undirected graph of order n and with eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$. Then the energy of G is defined as $E(G) = \sum_{k=1}^{n} |\lambda_k|$. This concept was given by Gutman (1978). For details related to graph energy see [12]. The concept of energy was extended to digraphs by Peña and Rada [20] and they defined the energy of a digraph D as $E(D) = \sum_{i=1}^{n} |\Re z_i|$, where $z_1, z_2, \ldots, z_n$ are eigenvalues of D, possibly complex and $\Re z_i$ denote the real part of complex number z_i. For more on the energy of digraphs see [7,23]. Recall that the trace norm of a complex matrix $B \in M_n(\mathbb{C})$ is defined as

$$\|B\|_* = \sum_{i=1}^{n} \sigma_i(B),$$

where $\sigma_1(B) \geq \sigma_2(B) \geq \sigma_3(B) \geq \cdots \geq \sigma_n(B)$ are the singular values of B i.e., the positive square roots of eigenvalues of BB^*. Throughout this paper. We call singular values of $A_\alpha(D)$ as the α singular values of D and trace norm of $A_\alpha(D)$ as the α trace norm of D and we will denote the trace norm of $A_\alpha(D)$ by $\|D_\alpha\|_*$. If a digraph D has k distinct singular values $\sigma_{1\alpha}, \sigma_{2\alpha}, \ldots, \sigma_{k\alpha}$, with their respective multiplicities $m_1, m_2, \ldots, m_k$, then we write the set of singular values as

$$\{\sigma_{1\alpha}^{[m_1]}, \sigma_{2\alpha}^{[m_2]}, \ldots, \sigma_{k\alpha}^{[m_k]}\}.$$

If $B = A(G)$, the adjacency matrix of graph G, then $\sigma_i(B) = |\lambda_i(G)|$ and so trace norm coincides with the energy of a graph. Trace norm of a matrix is also known as Nikiforov energy of a matrix [17]. So, graph energy extends to digraphs via trace norm as well.

Kharaghani and Tayfeh-Rezaie [11] obtained upper bounds on the trace norm of $(0,1)$-matrices. Agudelo and Rada [1] obtained lower bounds for the trace norm (Nikiforov's energy) of adjacency matrices of digraphs. Agudelo, Peña and Rada [2] determined trees attaining minimum and maximum trace norm. Monsalve and Rada [16] determined oriented bipartite graphs with minimum trace norm. Agudelo, Rada and Rivera [3] obtained upper bound for the trace norm of the Laplacian matrix of a digraph in terms of number of vertices n, number of arcs a and outdegrees of a digraph. For a digraph D with outdegree sequence $[d_1^+, d_2^+, \ldots, d_n^+]$ the first outdegree Zagreb index is defined as $Zg^+(D) = \sum_{i=1}^{n} (d_i^+)^2$. For more about $Zg^+(D)$ see [8]. For more about trace norm see [9,18,19,21].

The rest of the paper is organized as follows.

In section 2, we obtain the singular values of the A_α matrix of a directed path $\overrightarrow{P_n}$, with arc set $\mathcal{A}(\overrightarrow{P_n}) = \{(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n)\}$, directed cycle $\overrightarrow{C_n}$, with arc set $\mathcal{A}(\overrightarrow{C_n}) = \{(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1)\}$, and $\overrightarrow{K}_{r,s}$, where $\overrightarrow{K}_{r,s}$ denote oriented complete bipartite graph with partite sets $\{u_1, u_2, \ldots, u_r\}$, $\{v_1, v_2, \ldots, v_s\}$ and all arcs of the form (u_i, v_j), where $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, s$. We characterize digraphs with $Rank(A_\alpha(D)) = 1$. Using these results, we present a lower bound for the trace norm of $A_\alpha(D)$ of a digraph in terms of n, a and sum of squares of outdegrees of D and we characterize the digraphs attaining the lower bound. As a consequence, we determine oriented trees with minimum trace norm for A_α matrix.

In section 3, we obtain upper bounds for trace norm of A_α matrix of digraphs and characterize the extremal digraphs.

2. Lower bounds for trace norm of A_α matrix of digraphs

We first compute α-singular values of $\overrightarrow{P_n}, \overrightarrow{C_n}$ and $\overrightarrow{K}_{r,s}$, which will be used in our main results.

Lemma 2.1. *If $\overrightarrow{P_n}$ denote a directed path of order n, then the α-singular values of $\overrightarrow{P}_n$ are*

$$0^{[1]} \quad and \quad \sqrt{2\alpha^2 - 2\alpha + 1 + 2\alpha(1-\alpha)\cos\frac{j\pi}{n}},$$

where $j = 1, 2, \ldots, n-1$.

Proof. We have $A_\alpha(\overrightarrow{P_n}) = \alpha\Delta^+(\overrightarrow{P_n}) + (1-\alpha)A(\overrightarrow{P_n})$, so that

$$\begin{aligned} A_\alpha {A_\alpha}^T &= (\alpha\Delta^+ + (1-\alpha)A)(\alpha\Delta^+ + (1-\alpha)A)^T \\ &= (\alpha\Delta^+ + (1-\alpha)A)(\alpha\Delta^+ + (1-\alpha)A^T) \end{aligned}$$

$$
\begin{aligned}
&= \alpha^2(\Delta^+)^2 + \alpha(1-\alpha)\Delta^+ A^T + \alpha(1-\alpha)A\Delta^+ + (1-\alpha)^2 AA^T \\
&= \alpha^2(\Delta^+)^2 + \alpha(1-\alpha)(A\Delta^+ + \Delta^+ A^T) + (1-\alpha)^2 AA^T \\
&= (\alpha^2 + (1-\alpha)^2)\begin{bmatrix} I_{n-1} & \mathbf{0}_{n-1\times 1} \\ \mathbf{0}_{1\times n-1} & 0_{1\times 1}\end{bmatrix} + \alpha(1-\alpha)\begin{bmatrix} A(\overleftrightarrow{P_{n-1}}) & \mathbf{0}_{n-1\times 1} \\ \mathbf{0}_{1\times n-1} & 0_{1\times 1}\end{bmatrix}
\end{aligned}
$$

As the eigenvalues of path P_{n-1} are $2\cos\frac{\pi j}{n}$, where $j = 1, 2, \ldots, n-1$. Consequently, the eigenvalues of $A_\alpha {A_\alpha}^T$ are

$$0^{[1]} \text{ and } 2\alpha^2 - 2\alpha + 1 + 2\alpha(1-\alpha)\cos\frac{j\pi}{n},$$

where $j = 1, 2, \ldots, n-1$.
The singular values of A_α are

$$0^{[1]} \text{ and } \sqrt{2\alpha^2 - 2\alpha + 1 + 2\alpha(1-\alpha)\cos\frac{j\pi}{n}},$$

where $j = 1, 2, \ldots, n-1$. □

Lemma 2.2. *If $\overrightarrow{C_n}$ is a directed cycle of order n, then the α-singular values of $\overrightarrow{C_n}$ are*

$$\sqrt{(2\alpha^2 - 2\alpha + 1) + 2\alpha(1-\alpha)\cos\frac{2j\pi}{n}},$$

where $j = 0, 1, 2, 3, \ldots, n-1$.

Proof. We have $A_\alpha(\overrightarrow{C_n}) = \alpha\Delta^+(\overrightarrow{C_n}) + (1-\alpha)A(\overrightarrow{C_n})$, so that

$$
\begin{aligned}
A_\alpha {A_\alpha}^T &= (\alpha\Delta^+ + (1-\alpha)A)(\alpha\Delta^+ + (1-\alpha)A)^T \\
&= (\alpha I_n + (1-\alpha)A)(\alpha I_n + (1-\alpha)A^T) \\
&= \alpha^2 I_n + \alpha(1-\alpha)A^T + \alpha(1-\alpha)A + (1-\alpha)^2 AA^T \\
&= [\alpha^2 + (1-\alpha)^2]I_n + \alpha(1-\alpha)(A + A^{-1}),
\end{aligned}
$$

by using the fact that $AA^T = I_n$, for adjacency matrix A of directed cycle $\overrightarrow{C_n}$. We see that the eigenvalues of $A_\alpha {A_\alpha}^T$ are $(2\alpha^2 - 2\alpha + 1) + \alpha(1-\alpha)(\omega^j + \frac{1}{\omega^j})$, where $\omega^n = 1$ and $j = 0, 1, 2, \ldots, n-1$.
or $\sigma_j^2 = (2\alpha^2 - 2\alpha + 1) + 2\alpha(1-\alpha)\cos\frac{2j\pi}{n}$, where $j = 0, 1, 2, \ldots, n-1$.
or $\sigma_j = \sqrt{(2\alpha^2 - 2\alpha + 1) + 2\alpha(1-\alpha)\cos\frac{2j\pi}{n}}$, where $j = 0, 1, 2, \ldots, n-1$. □

Lemma 2.3. *If $\overrightarrow{K}_{r,s}$ be a complete bipartite digraph with partite sets $X = \{u_1, u_2, \ldots, u_r\}$ and $Y = \{v_1, v_2, \ldots, v_s\}$ and arcs (u_i, v_j) where $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, s$, then the α-singular values of $\overrightarrow{K}_{r,s}$ are*

$$0^{[s]},\ (\alpha s)^{[r-1]} \quad \textit{and} \quad \sqrt{\alpha^2 s^2 + (1-\alpha)^2 sr}$$

Proof. We have

$$A_\alpha(\overrightarrow{K}_{r,s}) = \begin{bmatrix} \alpha s I_r & (1-\alpha)J_{r\times s} \\ \mathbf{0}_{s\times r} & \mathbf{0}_{s\times s}\end{bmatrix}$$

so that

$$A_\alpha A_\alpha^T = \begin{bmatrix} \alpha^2 s^2 I_r + (1-\alpha)^2 s J_{r\times r} & \mathbf{0}_{r\times s} \\ \mathbf{0}_{s\times r} & \mathbf{0}_{s\times s}\end{bmatrix}$$

The eigenvalues of $A_\alpha {A_\alpha}^T$ are

$$0^{[s]},\ (\alpha^2 s^2)^{[r-1]} \quad \text{and} \quad \alpha^2 s^2 + (1-\alpha)^2 sr$$

Consequently, the singular values of A_α are

$$0^{[s]},\ (\alpha s)^{[r-1]} \text{ and } \sqrt{\alpha^2 s^2 + (1-\alpha)^2 sr}. \quad □$$

Remark 2.4. For $\alpha \in (0,1)$, $[A_\alpha(D)]^T$ need not be same as $A_\alpha(D^T)$, where D^T denotes transpose (or converse) of digraph D. So a digraph and its transpose need not have same α-singular values.

Recall that the rank of a matrix $B \in M_{m\times n}(\mathbb{C})$ is the number of nonzero singular values of B. Rank of a matrix is also defined as the maximum number of its linearly independent rows or columns. Recently rank of digraphs has been studied in [4,25]. The authors define rank preserving operations and characterize oriented graphs with rank 1, 2, 3. We characterize digraphs D such $\text{Rank}(A_\alpha(D)) = 1$ and this result will be used to discuss equality case in our main results. We give a different proof here, which will serve as an alternative proof to corresponding rank characterization for adjacency matrix of digraphs.

Theorem 2.5. *If D is a digraph of order $n \geq 2$, then $Rank(A_\alpha(D)) = 1$ if and only if $\alpha = 0$ and $D = \overrightarrow{K}_{r,s} \cup (n-r-s)K_1$ or $\alpha \neq 0$ and $D = \overrightarrow{K}_{1,s} \cup (n-s-1)K_1$, where $1 \leq s \leq n-1$ or $\alpha = \frac{1}{2}$ and $D = \overleftrightarrow{K_2}$.*

Proof. We prove this result when D has no isolated vertices, since adding isolated vertices does not change the rank. For $n=2$, it is easy to see that $\text{Rank}(A_\alpha(D)) = 1$, if and only if $\alpha = \frac{1}{2}$ and $D = \overleftrightarrow{K_2}$ or $\alpha \in [0\ 1)$ and $D = \overrightarrow{P_2}$. Assume $n \geq 3$ and $\text{Rank}(A_\alpha(D)) = 1$. We first claim that D cannot have two consecutive arcs of the form (v_1, v_2) and (v_2, v_3), where $X = \{v_1, v_2, v_3\} \subseteq \mathcal{V}(D)$. For if there are two consecutive arcs (v_1, v_2) and (v_2, v_3). Then D contains one of the eleven digraphs D_i, where $i = 1, 2, \ldots, 11$ shown in Fig. 1 as induced subdigraphs on vertex subset X.
If D contains D_1, D_2 *or* D_3, then since there can be arcs from X to $\mathcal{V} - X$, the principal submatrix of $A_\alpha(D)$ corresponding to vertices in X is of the form

$$P = \begin{bmatrix} \alpha d_1^+ & 1-\alpha & 0 \\ 0 & \alpha d_2^+ & 1-\alpha \\ 0 & 0 & \alpha d_3^+ \end{bmatrix}$$

with $d_1^+ \geq 1, d_2^+ \geq 1, d_3^+ \geq 0$ and $2 \leq d_1^+ + d_2^+ + d_3^+ \leq a$
if $\alpha \neq 0$, then $Rank(P) = 2$ or 3 according as $d_3^+ = 0$ or not.
Also, for $\alpha = 0$, $\text{Rank}(P) = 2$. Consequently, $\text{Rank}(A_\alpha(D))$ is at least 2 in this case, a contradiction.
If D contains one of induced subdigraphs D_4, D_5, D_6 *or* D_7, then the principal submatrix Q of $A_\alpha(D)$ corresponding to vertices in X has the form

$$Q = \begin{bmatrix} \alpha d_1^+ & 1-\alpha & 0 \\ 0 & \alpha d_2^+ & 1-\alpha \\ 1-\alpha & 0 & \alpha d_3^+ \end{bmatrix}$$

with $d_1^+, d_2^+, d_3^+ \geq 1$ and $3 \leq d_1^+ + d_2^+ + d_3^+ \leq a$
For $\alpha \in [0,1)$, $\text{Rank}(Q) = 3$. Consequently $\text{Rank}(A_\alpha(D))$ is at least three in this case, a contradiction.
Finally, if D contains one of the induced subdigraphs D_9, D_{10}, D_{11}, then the principal submatrix R of $A_\alpha(D)$ corresponding to vertices in X has the form

$$R = \begin{bmatrix} \alpha d_1^+ & 1-\alpha & 1-\alpha \\ 0 & \alpha d_2^+ & 1-\alpha \\ 0 & 0 & \alpha d_3^+ \end{bmatrix}$$

with $d_1^+ \geq 2, d_2^+ \geq 1, d_3^+ \geq 0$ and $3 \leq d_1^+ + d_2^+ + d_3^+ \leq a$
If $\alpha = 0$, then $\text{Rank}(R) = 2$. If $\alpha \neq 0$, then $\text{Rank}(R) = 2$ or 3 according as $d_3^+ = 0$ or $d_3^+ \geq 1$ Consequently, $\text{Rank}(A_\alpha(D))$ is at least 2 in this case, a contradiction.
Hence, D cannot have two consecutive arcs (v_1, v_2) and (v_2, v_3). This proves the claim. Therefore D is a sink source digraph i.e., each vertex of D is either a sink or a source vertex. By proposition 2.2 in [16] D is bipartite. Either D is $\overrightarrow{K}_{r,s}$ or a proper subdigraph of $\overrightarrow{K}_{r,s}$ with no isolated vertex.
Since α-matrix of $\overrightarrow{K}_{r,s}$ has the form

$$A_\alpha(\overrightarrow{K}_{r,s}) = \begin{bmatrix} \alpha s I_r & (1-\alpha) J_{r\times s} \\ 0_{s\times r} & 0_{s\times s} \end{bmatrix}$$

It is easy to see that any proper subdigraph of $\overrightarrow{K}_{r,s}$ with no isolated vertex has rank at least two. Hence only possibility is that $D = \overrightarrow{K}_{r,s}$. From Lemma 2.3, we see $Rank(A_\alpha(\overrightarrow{K}_{r,s})) = 1$ if and only if $\alpha = 0$ or $\alpha \neq 0$ and $r = 1$, by noting that rank of a matrix equals to number of its non zero singular values. □

Recall a digraph is said to be discrete if it has no arcs. Using the definition and singular value decomposition, the following result holds.

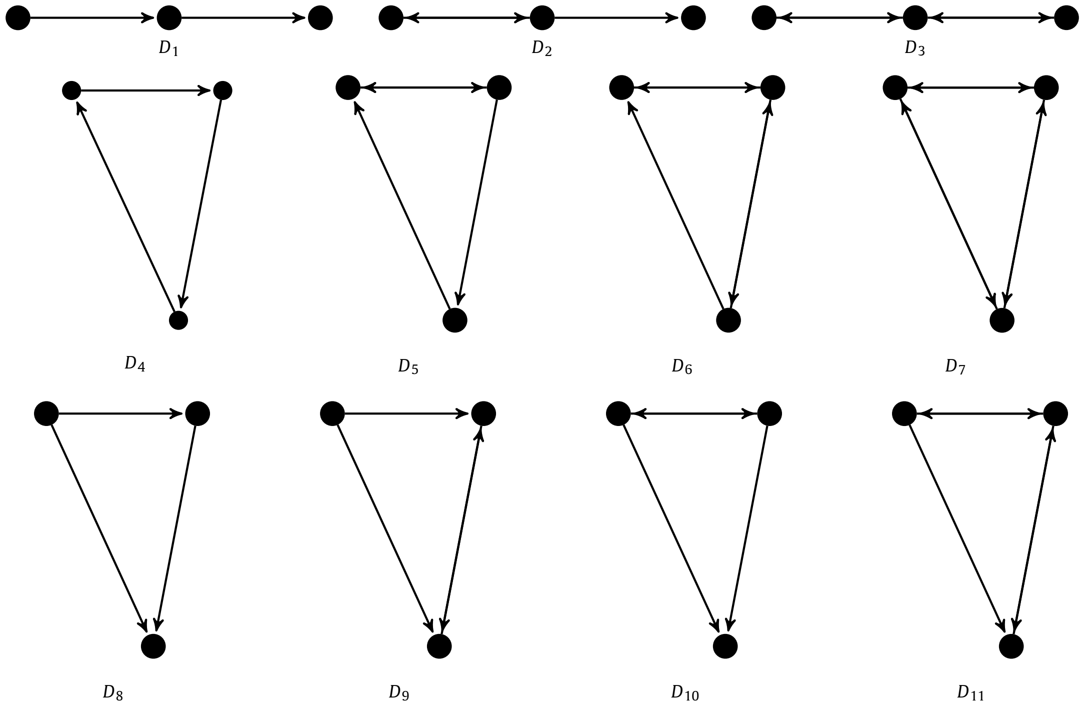

Fig. 1. Digraphs of order 3 with consecutive arcs and rank of $A_\alpha > 1$.

Lemma 2.6. *D is a discrete digraph of order n if and only if α-singular values of D are $0^{[n]}$.*

Let $D_1 = (\mathcal{V}_1, \mathcal{A}_1), D_2 = (\mathcal{V}_2, \mathcal{A}_2), \ldots, D_k = (\mathcal{V}_k, \mathcal{A}_k)$ be k digraphs. Then their direct sum $D = \oplus_{i=1}^{k} D_i$ is digraph with vertex set $\mathcal{V} = \cup_{i=1}^{k} \mathcal{V}_i$ and arc set $\mathcal{A} = \cup_{i=1}^{k} \mathcal{A}_i$. The following result is easy to prove.

Lemma 2.7. *If D is a direct sum of $D_1, D_2, \ldots, D_k$ digraphs, then*

$$\|D_\alpha\|_* = \sum_{i=1}^{k} \|(D_i)_\alpha\|_*.$$

Lemma 2.8. *If D is a k-regular digraph, then each row sum of ${A_\alpha A_\alpha}^T$ equals k^2.*

Proof. For a k-regular digraph

$$\begin{aligned} A_\alpha {A_\alpha}^T &= [\alpha k I_n + (1-\alpha)A][\alpha k I_n + (1-\alpha)A]^T \\ &= [\alpha k I_n + (1-\alpha)A][\alpha k I_n + (1-\alpha)A^T] \\ &= \alpha^2 k^2 I_n + k\alpha(1-\alpha)(A + A^T) + (1-\alpha)^2 A A^T. \end{aligned}$$

We first compute the row sum of $A_\alpha {A_\alpha}^T$. Note that the ith row sum of AA^T is equal to $\sum_{j=1}^{n} \langle R_i, R_j \rangle$. In each row R_i of A, there are k non-zero entries equal to 1. Assume the ones are at the positions $i_1, i_2, \cdots i_k$.
For each i_l where $1 \le l \le k$ as $d^-(v_{i_l}) = k$, there are $k-1$ ones above and below entry $[A]_{i,i_l}$.
Therefore,

$$\begin{aligned} \sum_{j=1}^{n} \langle R_i, R_j \rangle = \langle R_i, R_i \rangle + \sum_{j \neq i} \langle R_i, R_j \rangle \\ = k + k(k-1) \\ = k^2 \end{aligned}$$

Now,

$$\begin{aligned}\sum_{j=1}^{n}[A_\alpha {A_\alpha}^T]_{ij} &= \alpha^2k^2 + k\alpha(1-\alpha)(2k) + (1-\alpha)^2k^2\\ &= \alpha^2k^2 + 2k^2\alpha - 2k^2\alpha^2 + k^2 + k^2\alpha^2 - 2k^2\alpha\\ &= k^2. \quad \square\end{aligned}$$

We recall that if $\sigma_{1\alpha}, \sigma_{2\alpha}, \dots, \sigma_{n\alpha}$ are singular values of α matrix $A_\alpha(D)$ of a digraph D, then $\sigma_{1\alpha}$ is known as the α-spectral norm of D. Cruz, Giraldo and Rada [6] obtained a lower bound for the spectral norm of a digraph. We next determine a lower bound for α spectral norm, independent of α.

Lemma 2.9. *If D is a digraph with $n \geq 2$ vertices and a arcs, then $\sigma_{1\alpha}(D) \geq \frac{a}{n}$ with equality if and only if D is $\frac{a}{n}$-regular digraph.*

Proof. We recall $\|.\|_2$ denote usual Euclidean norm. Let $\mathbf{e} \in \mathbb{R}^n$ be the column vector with all entries 1. As $A_\alpha {A_\alpha}^T$ is real symmetric matrix, using Rayleigh quotient and Cauchy Schwarz inequality, we see

$$\begin{aligned}(\sigma_{1\alpha}(D))^2 &= \max_{x\neq 0}\frac{x^TA_\alpha A_\alpha^Tx}{x^Tx} = \max_{x\neq 0}\frac{x^TA_\alpha^TA_\alpha x}{x^Tx}\\ &= \max_{x\neq 0}\frac{\|A_\alpha x\|_2^2}{\|x\|_2^2} \geq \frac{\|A_\alpha e\|_2^2}{\|e\|_2^2}\\ &= \frac{\sum_{j=1}^{n}(d_j^+)^2}{n} \geq \frac{(\sum_{j=1}^{n}d_j^+)^2}{n^2} = \frac{a^2}{n^2}.\end{aligned}$$

Equality holds if and only if $d_j^+ = \text{constant} = d^+$, say for all $j = 1, 2, \cdots, n$ and $\mathbf{e}$ is an eigenvector of $A_\alpha^TA_\alpha$ i.e., row sums of $A_\alpha^TA_\alpha$ are constant.
As D is outregular (i.e., all outdegrees of D are equal), therefore $A_\alpha\mathbf{e} = d^+\mathbf{e}$.
Now row sums of $A_\alpha^TA_\alpha$ are constant gives $A_\alpha^T[d^+\mathbf{e}] = c\mathbf{e}$, where c is some constant.
or $\alpha(d^+)^2\mathbf{e} + (1-\alpha)d^+\mathbf{d} = c\mathbf{e}$, where $\mathbf{d} = [d_1^-, d_2^-, \dots, d_n^-]^T \in \mathbb{R}^n$ is indegree column vector of D.
In particular for $i \neq j$, we have

$$\alpha(d^+)^2 + (1-\alpha)d^+d_i^- = \alpha(d^+)^2 + (1-\alpha)d^+d_j^-,$$

which upon simplification gives $d_i^- = d_j^-$ for all $i, j = 1, 2, \dots, n$ and $i \neq j$. This implies D is inregular (i.e., all indegrees of D are equal) as well. Since

$$\sum_{j=1}^{n} d_j^+ = \sum_{j=1}^{n} d_j^-,$$

we see that $d_j^+ = d_j^- = \frac{a}{n}$ for all $j = 1, 2, \dots, n$, which in turn implies D is $\frac{a}{n}$-regular digraph. $\square$

We next obtain a lower bound for the α trace norm of digraphs.

Theorem 2.10. *Let D be a digraph with $n \geq 2$ vertices and a arcs and $A_\alpha = \alpha\Delta^+ + (1-\alpha)A$ be the α matrix of D and $\sigma_{1\alpha} \geq \sigma_{2\alpha} \geq \cdots \geq \sigma_{n\alpha} \geq 0$ be the α-singular values of D. Then*

$$\|D_\alpha\|_* \geq \sqrt{(1-\alpha)^2a + \alpha^2\sum_{i=1}^{n}(d_i^+)^2 + n(n-1)|detA_\alpha|^{\frac{2}{n}}}, \tag{2.1}$$

with equality if and only if
(a) D is a Discrete digraph (b) $\alpha = 0$ and D is the direct sum of directed cycles.
(c) α=0 and $D = \overrightarrow{K}_{r,s}$ + Possibly some isolated vertices or $\alpha \neq 0$, $D = \overrightarrow{K}_{1,s}$ + Possibly some isolated vertices, where $1 \leq s \leq n-1$ or $\alpha = \frac{1}{2}$ and $D = \overleftrightarrow{K_2}$.

Proof. Let $\sigma_{1\alpha} \geq \sigma_{2\alpha} \geq, \cdots \geq \sigma_{n\alpha} \geq 0$ be the singular values of $A_\alpha = \alpha\Delta^+ + (1-\alpha)A$. By AM-GM inequality

$$\|D_\alpha\|_*^2 = (\sum_{i=1}^{n}\sigma_{i\alpha})^2$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} (\sigma_{i\alpha})^2 + 2 \sum_{1 \le i < j \le n} \sigma_{i\alpha}\sigma_{j\alpha} \\
&= (1-\alpha)^2 \sum_{i=1}^{n} d_i^+ + \alpha^2 \sum_{i=1}^{n} (d_i^+)^2 + \frac{2n(n-1)}{n(n-1)} \sum_{1 \le i < j \le n} \sigma_{i\alpha}\sigma_{j\alpha} \\
&\ge (1-\alpha)^2 a + \alpha^2 \sum_{i=1}^{n} (d_i^+)^2 + n(n-1)(\prod_{i=1}^{n} (\sigma_{i\alpha})^{n-1})^{\frac{2}{n(n-1)}} \\
&= (1-\alpha)^2 a + \alpha^2 \sum_{i=1}^{n} (d_i^+)^2 + n(n-1)|det A_\alpha|^{\frac{2}{n}}
\end{aligned}
$$

$$
\|D_\alpha\|_* \ge \sqrt{(1-\alpha)^2 a + \alpha^2 \sum_{i=1}^{n} (d_i^+)^2 + n(n-1)|det A_\alpha|^{\frac{2}{n}}}
$$

This proves (2.1).
Assume equality holds in (2.1), then equality holds in AM-GM inequality which gives

$$
\sigma_{i\alpha}\sigma_{j\alpha} = c
$$

for all i, j. Three cases depending on α- singular values arise here
Case (1). If $\sigma_{1\alpha} = 0$, then $\sigma_{1\alpha} = \sigma_{2\alpha} = \ldots \sigma_{3\alpha} = 0$. This implies D is a Discrete digraph, by Lemma 2.6.
Case (2). If $\sigma_{1\alpha} > 0, \sigma_{2\alpha} > 0$. Then $\sigma_{1\alpha} = \sigma_{2\alpha} = \cdots = \sigma_{n\alpha}$. By singular value decomposition, there exists a real orthogonal matrix U such that

$$
\begin{aligned}
& A_\alpha = \sigma_{1\alpha} U \\
\text{or} \quad & [A_\alpha]_{ij} = \sigma_{1\alpha}[U]_{i,j} \\
\text{or} \quad & \sum_{j=1}^{n} [A_\alpha]_{ij}^2 = (\sigma_{1\alpha})^2 \sum_{j=1}^{n} [U]_{ij}{}^2 = (\sigma_{1\alpha})^2 \\
\text{or} \quad & \alpha^2 (d_i^+)^2 + (1-\alpha)^2 d_i^+ = (\sigma_{1\alpha})^2 \text{ for all } i = 1, 2, 3, \ldots, n
\end{aligned}
$$

Again

$$
\sum_{i=1}^{n} [A_\alpha]_{ij}^2 = (\sigma_{1\alpha})^2 \sum_{i=1}^{n} [U]_{ij}{}^2
$$

$$
\text{or} \quad \alpha^2 (d_j^+)^2 + (1-\alpha)^2 d_j^- = (\sigma_{1\alpha})^2 \quad \text{for all } j = 1, 2, 3, \ldots, n
$$

In particular

$$
\alpha^2 (d_i^+)^2 + (1-\alpha)^2 d_i^+ = \alpha^2 (d_i^+)^2 + (1-\alpha)^2 d_i^-
$$

which gives $d_i^+ = d_i^- = d$, say for all $i = 1, 2, \ldots, n$. This implies D is d-regular.
Using Lemma 2.9, we see,

$$
\begin{aligned}
\sum_{i=1}^{n} (\sigma_{i\alpha})^2 &= (1-\alpha)^2 a + \alpha^2 \sum_{i=1}^{n} (d_i^+)^2 \\
n(\sigma_{i\alpha})^2 &= (1-\alpha)^2 nd + \alpha^2 nd^2 \\
nd^2 &= (1-\alpha)^2 nd + \alpha^2 nd^2 \\
(1-\alpha^2) nd^2 &= (1-\alpha)^2 nd \\
d &= \frac{1-\alpha}{1+\alpha}
\end{aligned}
$$

which gives

$$
\alpha = 0 \quad \textit{and} \;\; d = 1
$$

So, D is direct sum of directed cycles and $\alpha = 0$.
Case (3). If $\sigma_{1\alpha} > 0, \sigma_{2\alpha} = 0$, then $\sigma_{1\alpha} > 0, \sigma_{2\alpha} = \sigma_{3\alpha} = \ldots \sigma_{n\alpha} = 0$.
By singular value decomposition, there exists real orthogonal matrices $U = (u_{ij})_{n\times n}$ and $V = (v_{ij})_{n\times n}$ such that

$$A_\alpha = USV^T,$$

where $S = diag(\sigma_{1\alpha}, 0, 0, \ldots, 0) = diag(\sqrt{c}, 0, 0, \ldots, 0)$, where $c = (1-\alpha)^2 a + \alpha^2 \sum_{i=1}^{n} (d_i^+)^2$
In this case $A_\alpha = \sqrt{c} u v^T$, where $u = [u_{11}, \quad u_{21}, \quad \ldots, u_{n1}]^T$ and $v = [v_{11}, \quad v_{21}, \quad \ldots, v_{n1}]^T$.
This implies A_α is a rank 1 matrix. By Theorem 2.5, we see that $\alpha = 0$ and $D = \overrightarrow{K}_{r,s}$ + Possibly some isolated vertices or $\alpha \neq 0$, $D = \overrightarrow{K}_{1,s}$, where $1 \leq s \leq n-1$ or $\alpha = \frac{1}{2}$ and $D = \overleftrightarrow{K_2}$.

Conversely, if D is discrete digraph, then both sides of (2.1) are equal to zero. If $\alpha = 0$ and D is direct sum of cycles, then both sides of (2.1) are equal to a, the number of arcs in D. If $\alpha = 0$ and $D = \overrightarrow{K}_{r,s}$ + Possibly some isolated vertices, then both sides of (2.1) are equal to $\sqrt{rs}$. If $\alpha = \frac{1}{2}$ and $D = \overleftrightarrow{K_2}$, then both sides of (2.1) are equal to 1. If $\alpha \neq 0$ and $D = \overrightarrow{K}_{1,s}$ + Possibly some isolated vertices then both sides of (2.1) are also equal. This completes the proof. □

Following is an immediate consequence of Theorem 2.10.

Corollary 2.11. *If D is a digraph with $n \geq 2$ vertices and a arcs, then*

$$\|D_\alpha\|_* \geq \sqrt{(1-\alpha)^2 a + \alpha^2 \sum_{i=1}^{n} (d_i^+)^2},$$

with equality if and only if D satisfies one of the following
(a) D is a discrete digraph.
(b) $\alpha = 0$ and $D = \overrightarrow{K}_{r,s}$ + Possibly some isolated vertices or $\alpha \neq 0$, $D = \overrightarrow{K}_{1,s}$ + Possibly some isolated vertices, where $1 \leq s \leq n-1$ or $\alpha = \frac{1}{2}$ and $D = \overleftrightarrow{K_2}$.

Given a tree T on n, vertices, let $\mathcal{T}(n)$ denote the set of oriented trees with underlying tree T. The next result determines oriented trees having minimum α-trace norm in $\mathcal{T}(n)$.

Corollary 2.12. *If $T \in \mathcal{T}(n)$, then*

$$\|T_\alpha\|_* \geq \sqrt{(1-\alpha)^2 (n-1) + \alpha^2 \sum_{i=1}^{n} (d_i^+)^2}, \tag{2.2}$$

with equality if and only if $\alpha = 0$ and $T = \overrightarrow{K}_{1,n-1}$ or $\overrightarrow{K}_{n-1,1}$ or $\alpha \neq 0$, $T = \overrightarrow{K}_{1,n-1}$.

Proof. We note that for a digraph D

$$\|D_\alpha\|_* \geq \sqrt{(1-\alpha)^2 a + \alpha^2 \sum_{i=1}^{n} (d_i^+)^2}$$

with equality if and only if $\alpha = 0$ and $D = \overrightarrow{K}_{r,s}$ + Possibly some isolated vertices or $\alpha \neq 0$, $D = \overrightarrow{K}_{1,s}$ + Possibly some isolated vertices, where $1 \leq s \leq n-1$ or $\alpha = \frac{1}{2}$ and $D = \overleftrightarrow{K_2}$.
For an oriented tree $T \in \mathcal{T}(n)$, we see

$$\|T_\alpha\|_* \geq \sqrt{(1-\alpha)^2 (n-1) + \alpha^2 \sum_{i=1}^{n} (d_i^+)^2}.$$

Also, since T has no isolated vertices and no cycles and $\overrightarrow{K}_{r,s}$ is a tree if and only if either $r = 1$ and $s = n-1$ or $r = n-1$ and $s = 1$, the result follows. □

3. Upper bounds for trace norm of A_α matrix of digraphs

The well known McClelland's upper bound [Theorem 5.1, [12]] states that for a graph G with p vertices and q edges

$$E(G) \leq \sqrt{2pq}. \tag{3.1}$$

Moreover, equality holds in (3.1) if and only if G is a discrete graph or $G = (\frac{p}{2})K_2$.
This upper bound has been extended to digraph energy by Rada [22]. We next find McClelland type upper bound for trace norm of A_α matrix of a digraph. Recall that for nonnegative real numbers $x_1, x_2, \ldots, x_n$, their variance $var(x_1, x_2, \ldots, x_n) \geq 0$ with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Theorem 3.1. *Let D be a digraph with n vertices, a arcs and let $(d_1^+, d_2^+, \ldots, d_n^+)$ be outdegrees of vertices. Then*

$$\|D\|_* \leq \sqrt{n[(1-\alpha)^2 a + \alpha^2 \sum_{i=1}^{n} (d_i^+)^2]}, \tag{3.2}$$

with equality if and only if D satisfies one of the following
(a) D is a Discrete digraph.
(b) $\alpha = 0$ and D is a direct sum of directed cycles.

Proof. Let $\sigma_{1\alpha} \geq \sigma_{2\alpha} \geq, \cdots \geq \sigma_{n\alpha} \geq 0$ be the singular values of $A_\alpha = \alpha\Delta^+ + (1-\alpha)A$
We know that

$$\begin{aligned}
&0 \leq var[\sigma_{1\alpha}, \sigma_{2\alpha}, \sigma_{3\alpha}, \ldots, \sigma_{n\alpha}] \\
&= \frac{1}{n}\sum_{i=1}^{n}(\sigma_{i\alpha})^2 - (\frac{\sum_{i=1}^{n}\sigma_{i\alpha}}{n})^2 \\
&= \frac{1}{n}[(1-\alpha)^2 a + \alpha^2 \sum_{i=1}^{n}(d_i^+)^2] - \frac{\|D_\alpha\|_*^2}{n^2} \\
\text{or} \quad & \frac{\|D_\alpha\|_*^2}{n^2} \leq \frac{1}{n}[(1-\alpha)^2 a + \alpha^2 \sum_{i=1}^{n}(d_i^+)^2] \\
\text{or} \quad & \frac{\|D_\alpha\|_*^2}{n} \leq [(1-\alpha)^2 a + \alpha^2 \sum_{i=1}^{n}(d_i^+)^2] \\
\text{or} \quad & \|D_\alpha\|_*^2 \leq n[(1-\alpha)^2 a + \alpha^2 \sum_{i=1}^{n}(d_i^+)^2] \\
\text{or} \quad & \|D_\alpha\|_* \leq \sqrt{n[(1-\alpha)^2 a + \alpha^2 \sum_{i=1}^{n}(d_i^+)^2]}
\end{aligned}$$

This proves the inequality. The equality holds in (3.2) if and only if $\sigma_{1\alpha} = \sigma_{2\alpha} = \cdots = \sigma_{n\alpha}$. Proceeding as in the equality case of Theorem 2.10, we see equality holds if and only if D is a discrete digraph or $\alpha = 0$ and D is direct sum of cycles. □

Remark 3.2. If $D = \overleftrightarrow{G}$ is a symmetric digraph on n vertices, then for $\alpha = 0$, upper bound (3.2) reduces to McClelland's upper bound for graphs. Therefore, Theorem 3.1, is an extension of McClelland's inequality for graphs.

A square $(0, 1)$ matrix M of order n is the incidence matrix of a symmetric (n, k, λ)-BIBD if and only if

$$MM^T = \lambda J + (k-\lambda)I,$$

where J is matrix of all ones and I stands for identity matrix. If adjacency matrix of a digraph D satisfies this condition, we say D is a symmetric (n, k, λ)-BIBD.

A k-regular graph G of order n is said to be strongly regular graph (SRG) with parameters (n, k, λ, μ) if any two adjacent vertices of G have λ common neighbours and any two non adjacent vertices of G have μ common neighbours. For example Shirkhande graph is a strongly regular graph SRG $(16, 6, 2, 2)$. Since its two parameters λ *and* μ are equal, its adjacency matrix clearly satisfies condition for being a symmetric BIBD.

We next obtain Koolen and Moulton type upper bound [10] for the α trace norm of digraphs. We adapt the idea from [17] and discuss the equality case. Here by $d_{\max}^+$, we denote the maximum outdegree of digraph D.

Theorem 3.3. *Let D be a digraph with $n \geq 2$ vertices and $a \geq n\beta$ arcs, where $\beta = \max(1-\alpha, \alpha\, d^{+}_{max})$. Then*

$$\|D_\alpha\|_* \leq \frac{a}{n} + \sqrt{(n-1)[(1-\alpha)^2 a + \alpha^2 \sum_{i=1}^{n} (d_i^+)^2 - \frac{a^2}{n^2}]} \tag{3.3}$$

with equality if and only if D satisfies one of the following
(a) $\alpha = 0$ and D is direct sum of cycles. (b) D is $\frac{a}{n}$-regular with two distinct nonnegative singular values and these are $\sigma_{1\alpha} = \frac{a}{n}$, $\sigma_{2\alpha} = \sigma_{3\alpha} = \cdots = \sigma_{n\alpha} = \sigma = \sqrt{\frac{\alpha^2 \sum_{i=1}^{n} (d_i^+)^2 + (1-\alpha)^2 a - \frac{a^2}{n^2}}{n-1}}$.

Proof. Let $\sigma_{1\alpha}, \sigma_{2\alpha}, \ldots, \sigma_{n\alpha}$ be the α-singular values of D. Then by Cauchy Schwarz Inequality, we have

$$\begin{aligned}\|D_\alpha\|_* &= \sum_{i=1}^{n} \sigma_{i\alpha} = \sigma_{1\alpha} + \sum_{i=2}^{n} \sigma_{i\alpha} \\ &\leq \sigma_{1\alpha} + \sqrt{(n-1)\sum_{i=2}^{n} (\sigma_{i\alpha})^2} \\ &= \sigma_{1\alpha} + \sqrt{(n-1)[(1-\alpha)^2 a + \alpha^2 \sum_{i=2}^{n} (d_i^+)^2 - (\sigma_{1\alpha})^2]}\end{aligned}$$

Consider the function

$$f(x) = x + \sqrt{(n-1)[(1-\alpha)^2 a + \alpha^2 \sum_{i=2}^{n} (d_i^+)^2 - x^2]}$$

on $\left[0, \sqrt{(n-1)\left[(1-\alpha)^2 a + \alpha^2 \sum_{i=1}^{n} (d_i^+)^2\right]}\,\right]$. It is easy to verify that f is strictly decreasing on

$$\left[\sqrt{\frac{(1-\alpha)^2 a + \alpha^2 \sum_{i=1}^{n} (d_i^+)^2}{n}}, \sqrt{(1-\alpha)^2 a + \alpha^2 \sum_{i=1}^{n} (d_i^+)^2)}\,\right].$$

Also, with $a \geq n\beta$, we have

$$\begin{aligned}(1-\alpha)^2 a + \alpha^2 \sum_{i=1}^{n} (d_i^+)^2 &= Trace(A_\alpha A_\alpha^*) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} [A_\alpha]_{ij}^2 \\ &\leq \beta \sum_{i=1}^{n}\sum_{j=1}^{n} [A_\alpha]_{ij} = \beta a\end{aligned}$$

$$\text{or} \quad (1-\alpha)^2 a + \alpha^2 \sum_{i=1}^{n} (d_i^+)^2 \leq \beta a \leq \frac{a^2}{n}$$

$$\text{or} \quad \frac{(1-\alpha)^2 a + \alpha^2 \sum_{i=1}^{n} (d_i^+)^2}{n} \leq \frac{a^2}{n^2}$$

$$\text{or} \quad \sqrt{\frac{(1-\alpha)^2 a + \alpha^2 \sum_{i=1}^{n} (d_i^+)^2}{n}} \leq \frac{a}{n} \leq \sigma_{1\alpha} \quad \text{[By Lemma 2.9]}$$

Therefore,

$$\|D_\alpha\|_* \leq f(\sigma_{1\alpha}) \leq f(\frac{a}{n})$$

which proves bound (3.3).
Moreover, equality holds if and only if D has at most two singular values $\sigma_{1\alpha}, \sigma_{2\alpha} = \sigma_{3\alpha} = \cdots = \sigma_{n\alpha}$ and D is $\frac{a}{n}$-regular so $\sigma_{1\alpha} = \frac{a}{n}$.
In case D has only one singular value then $\sigma_{1\alpha} = \sigma_{2\alpha} = \cdots = \sigma_{n\alpha}$. Then since $a \geq n\beta$, proceeding as in equality case of Theorem 2.10, $\alpha = 0$ and D is direct sum of cycles. In view of Theorem 2.5, we see that if $\sigma_{i\alpha} = 0$, for $i = 2, 3, \ldots, n$, then $n = 2$, $\alpha = \frac{1}{2}$ and $D = \overleftrightarrow{K_2}$. Thus, we conclude that if D has two distinct positive singular values, then these are $\sigma_{1\alpha} = \frac{a}{n}, \sigma_{2\alpha} = \sigma_{3\alpha} = \ldots, \sigma_{n\alpha} = \sigma = \sqrt{\frac{\alpha^2 \sum_{i=1}^{n} (d_i^+)^2 + (1-\alpha)^2 a - \frac{a^2}{n^2}}{n-1}}$. Converse part can be easily verified. □

Example 3.4. It is easy to see that for $\alpha \in [0, 1)$, the singular values of $\overleftrightarrow{K}_n$ are $n-1, |n\alpha - 1|^{[n-1]}$. So, for α matrix of $\overleftrightarrow{K}_n$, equality holds for the upper bound given in Theorem 3.3.

Remark 3.5. At this moment, we are not able to determine digraphs for which A_α matrix has two distinct positive singular values of the form stated in Theorem 3.3. It remains a future problem to determine such digraphs. Here we note that for $\alpha = 0$, if a digraph D attain upper bound given in Theorem 3.3, then D must be a symmetric $(n, \frac{a}{n}, \frac{a(a-n)}{n^2(n-1)})$-BIBD. For example see [Theorem 1.1, [1]]. For other values of α, these digraphs need not have the desired property. For example take symmetric digraph associated with Shirkhande graph which is a strongly regular graph SRG $(16, 6, 2, 2)$. Its eigenvalues are $6, 2^{[6]}, -2^{[9]}$. The singular values of its α matrix are $6, (2+4\alpha)^{[6]}, |8\alpha - 2|^{[9]}$. We observe that for $\alpha = 0$, it has two distinct singular values.

Declaration of competing interest

The authors declare that they do not have any conflict of interest.

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Data availability

No data was used for the research described in the article.

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